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A GENERALIZATION OF SYMMETRIC AND SKEW-SYMMETRIC DETERMINANTS.

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1. Examples of symmetric and skew-symmetric determinants are, respectively,

$$Y=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}, \quad K=\begin{vmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{vmatrix}.$$

A direct generalization is afforded by the following determinant:

$$\Delta=\begin{vmatrix} a_{11} & a_{12}+iA_{12} & a_{13}+iA_{13} \\ a_{12}-iA_{12} & a_{22} & a_{23}+iA_{23} \\ a_{13}-iA_{13} & a_{23}-iA_{23} & a_{33} \end{vmatrix}.$$

For determinants like Δ, and D below, with a_{ij} and A_{ij} real, we have the

Theorem. *A determinant with symmetrically conjugate elements is real.*

For proof we reflect Δ on the main diagonal and obtain an equal determinant whose elements are conjugate with the corresponding ones of the given determinant. From $\Delta=\bar{\Delta}$ follows $\Delta=$real.

2. Expanding Δ, we get $\Delta=Y+Q$, where

$$Q\equiv -a_{11}A^2_{23}-a_{22}A^2_{13}-a_{33}A^2_{12}+2a_{12}A_{13}A_{23}+2a_{23}A_{12}A_{13}-2a_{13}A_{12}A_{23}.$$

Regarding A_{12}, A_{13}, and A_{23} as variables, we seek the conditions under which Q shall factor into linear expressions. To fix the ideas, let $a_{11}\neq 0$. Completing the square in A_{23}, we get finally

$$-a_{11}Q=(a_{11}A_{23}-a_{12}A_{13}+a_{13}A_{12})^2-A^2_{13}(a^2_{12}-a_{11}a_{22})$$
$$-A^2_{12}(a^2_{13}-a_{11}a_{33})-2A_{12}A_{13}(a_{11}a_{23}-a_{12}a_{13}).$$

The last three terms, with their signs changed, form a perfect square if and only if

$$(a^2_{12}-a_{11}a_{22})(a^2_{13}-a_{11}a_{33})-(a_{11}a_{23}-a_{12}a_{13})^2=0,$$

that is, if $a_{11}Y=0$. For $Y=0$ we therefore have

$$-a_{11}Q=(P+T)(P-T),\quad P=a_{11}A_{23}-a_{12}A_{13}+a_{13}A_{12},$$
$$T=A_{13}\sqrt{(a^2_{12}-a_{11}a_{22})}+A_{12}\sqrt{(a^2_{13}-a_{11}a_{33})},$$

the signs of the radicals being suitably chosen.

Interpreting the A_{ij} as homogeneous point-coördinates in the plane, we conclude that the equation $\triangle=Y$ represents a pair of straight lines if and only if $Y=0$.

3. Consider next the symmetrically conjugate determinant

$$D=\begin{vmatrix} a_{11} & a_{12}+iA_{12} & a_{13}+iA_{13} & a_{14}+iA_{14} \\ a_{12}-iA_{12} & a_{22} & a_{23}+iA_{23} & a_{24}+iA_{24} \\ a_{13}-iA_{13} & a_{23}-iA_{23} & a_{33} & a_{24}+iA_{34} \\ a_{14}-iA_{14} & a_{24}-iA_{24} & a_{34}-iA_{34} & a_{44} \end{vmatrix},$$

and the included symmetric and skew-symmetric determinants

$$a=|\,a_{ij}\,|,\quad A=\begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & 0 \end{vmatrix}.$$

By the theory of skew-symmetric determinants,

$$K\equiv 0,\quad A\equiv(A_{12}A_{34}-A_{13}A_{24}+A_{14}A_{23})^2.$$

To expand D, we write $a_{ii}=a_{ii}+0$, thus obtaining each element expressed as an algebraic sum of two terms. Hence D equals the sum of 16 determinants of the fourth order, among which occur a and A. In view ot the theorem of §1, 8 of the determinants cancel, namely, those in which the elements of an *odd* number of columns contain the factor i. Each of the remaining 6 determinants have the factor i^2. Their sum is

$$-\begin{vmatrix} a_{11} & a_{12} & A_{13} & A_{14} \\ a_{12} & a_{22} & A_{23} & A_{24} \\ a_{13} & a_{23} & 0 & A_{34} \\ a_{14} & a_{24} & -A_{34} & 0 \end{vmatrix} - \begin{vmatrix} a_{11} & A_{12} & A_{13} & a_{14} \\ a_{12} & 0 & A_{23} & a_{24} \\ a_{13} & -A_{23} & 0 & a_{34} \\ a_{14} & -A_{24} & -A_{34} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & A_{12} & a_{13} & A_{14} \\ a_{12} & 0 & a_{23} & A_{24} \\ a_{13} & -A_{23} & a_{33} & A_{34} \\ a_{14} & -A_{24} & a_{34} & 0 \end{vmatrix}$$

$$-\begin{vmatrix} 0 & a_{12} & a_{13} & A_{14} \\ -A_{12} & a_{22} & a_{23} & A_{24} \\ -A_{13} & a_{23} & a_{33} & A_{34} \\ -A_{14} & a_{24} & a_{34} & 0 \end{vmatrix} - \begin{vmatrix} 0 & a_{12} & A_{13} & a_{14} \\ -A_{12} & a_{22} & A_{23} & a_{24} \\ -A_{13} & a_{23} & 0 & a_{34} \\ -A_{14} & a_{24} & -A_{34} & a_{44} \end{vmatrix} - \begin{vmatrix} 0 & A_{12} & a_{13} & a_{14} \\ -A_{12} & 0 & a_{23} & a_{24} \\ -A_{13} & -A_{23} & a_{33} & a_{34} \\ -A_{14} & -A_{24} & a_{34} & a_{44} \end{vmatrix}$$

To the first of the six we apply Laplace's expansion, the dividing line separating the first two *columns* from the last two. For the other five, we first interchange columns to bring the a_{ij} into the first and second columns and then apply Laplace's expansion. In their final sum, the coefficient of $-A_{12}A_{34}$ is

$$\begin{vmatrix} a_{12} & a_{14} \\ a_{23} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{13} \\ a_{24} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{23} \\ a_{14} & a_{34} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{24} \\ a_{13} & a_{34} \end{vmatrix} = 2\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$$

Similar simplifications may be made in the coefficients of $A_{13}A_{24}$ and $A_{14}A_{23}$. A term A_{ij}^2 occurs singly. A term $A_{ij}A_{kl}$ in which two and but two of the subscripts $i, j\ k, l$ are equal occurs twice, with equal coefficients. We obtain the final result $D=a+A-S$, where (denoting a_{ij} by ij)

$$S=A^2_{12}\begin{vmatrix} 33 & 34 \\ 34 & 44 \end{vmatrix} + A^2_{13}\begin{vmatrix} 22 & 24 \\ 24 & 44 \end{vmatrix} + A^2_{14}\begin{vmatrix} 22 & 23 \\ 23 & 33 \end{vmatrix} + A^2_{23}\begin{vmatrix} 11 & 14 \\ 14 & 44 \end{vmatrix}$$

$$+A^2_{24}\begin{vmatrix} 11 & 13 \\ 13 & 33 \end{vmatrix} + A^2_{34}\begin{vmatrix} 11 & 12 \\ 12 & 22 \end{vmatrix} + 2A_{12}A_{34}\begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} + 2A_{13}A_{24}\begin{vmatrix} 12 & 14 \\ 23 & 34 \end{vmatrix}$$

$$+2A_{14}A_{23}\begin{vmatrix} 12 & 13 \\ 24 & 34 \end{vmatrix} - 2A_{12}A_{13}\begin{vmatrix} 23 & 24 \\ 34 & 44 \end{vmatrix} + 2A_{12}A_{14}\begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix}$$

$$-2A_{12}A_{24}\begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} - 2A_{13}A_{14}\begin{vmatrix} 22 & 24 \\ 23 & 34 \end{vmatrix} - 2A_{13}A_{23}\begin{vmatrix} 12 & 14 \\ 24 & 44 \end{vmatrix}$$

$$-2A_{13}A_{34}\begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} - 2A_{14}A_{24}\begin{vmatrix} 12 & 13 \\ 23 & 33 \end{vmatrix} + 2A_{14}A_{34}\begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix}$$

$$-2A_{23}A_{24}\begin{vmatrix} 11 & 13 \\ 14 & 34 \end{vmatrix} - 2A_{24}A_{34}\begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix} + 2A_{12}A_{23}\begin{vmatrix} 13 & 14 \\ 34 & 44 \end{vmatrix}$$

$$+2A_{23}A_{34}\begin{vmatrix} 11 & 14 \\ 12 & 24 \end{vmatrix}$$

A law for the simplified expansion of D, which may be readily extended to like determinants of any order, may now be given. The law may be expressed very luminously by use of a "multiplication table." As in the usual notation for quadratic forms, a term $2cxy$ is conveniently expressed by $cxy+cyx$. Then S is given by the accompanying multiplication table.

$$\begin{array}{c|cccccc}
 & A_{34} & A_{24} & A_{23} & A_{14} & A_{13} & A_{12} \\
\hline
A_{34} & \begin{vmatrix} 11 & 12 \\ 12 & 22 \end{vmatrix} & -\begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix} & \begin{vmatrix} 11 & 14 \\ 12 & 24 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix} & -\begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} \\
A_{24} & -\begin{vmatrix} 11 & 13 \\ 12 & 23 \end{vmatrix} & \begin{vmatrix} 11 & 13 \\ 13 & 33 \end{vmatrix} & -\begin{vmatrix} 11 & 14 \\ 13 & 34 \end{vmatrix} & -\begin{vmatrix} 12 & 13 \\ 23 & 33 \end{vmatrix} & \begin{vmatrix} 12 & 14 \\ 23 & 34 \end{vmatrix} & -\begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} \\
A_{23} & \begin{vmatrix} 11 & 12 \\ 14 & 24 \end{vmatrix} & -\begin{vmatrix} 11 & 13 \\ 14 & 34 \end{vmatrix} & \begin{vmatrix} 11 & 14 \\ 14 & 44 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 24 & 34 \end{vmatrix} & -\begin{vmatrix} 12 & 14 \\ 24 & 44 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 34 & 44 \end{vmatrix} \\
A_{14} & -\begin{vmatrix} 12 & 22 \\ 13 & 23 \end{vmatrix} & -\begin{vmatrix} 12 & 23 \\ 13 & 33 \end{vmatrix} & \begin{vmatrix} 12 & 24 \\ 13 & 34 \end{vmatrix} & \begin{vmatrix} 22 & 23 \\ 23 & 33 \end{vmatrix} & -\begin{vmatrix} 22 & 24 \\ 23 & 34 \end{vmatrix} & \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} \\
A_{13} & -\begin{vmatrix} 12 & 22 \\ 14 & 24 \end{vmatrix} & \begin{vmatrix} 12 & 23 \\ 14 & 34 \end{vmatrix} & -\begin{vmatrix} 12 & 24 \\ 14 & 44 \end{vmatrix} & -\begin{vmatrix} 22 & 23 \\ 24 & 34 \end{vmatrix} & \begin{vmatrix} 22 & 24 \\ 34 & 44 \end{vmatrix} & -\begin{vmatrix} 23 & 24 \\ 34 & 44 \end{vmatrix} \\
A_{12} & \begin{vmatrix} 13 & 23 \\ 14 & 24 \end{vmatrix} & -\begin{vmatrix} 13 & 33 \\ 14 & 34 \end{vmatrix} & \begin{vmatrix} 13 & 34 \\ 14 & 44 \end{vmatrix} & \begin{vmatrix} 23 & 33 \\ 24 & 34 \end{vmatrix} & -\begin{vmatrix} 23 & 34 \\ 24 & 44 \end{vmatrix} & \begin{vmatrix} 33 & 34 \\ 34 & 44 \end{vmatrix}
\end{array}$$

The law of formation is now evident. The body of the table gives in proper position and with proper sign the minors of $| a_{ij} | \equiv a$. It follows (Muir, *Theory of Determinants*, §174) that the determinant of the sixth order defined by the table equals a^3, being the second compound of a.

4. Changing the notation by replacing A_{13} and A_{24} by their negatives, we obtain from S a quadratic form S' with the signs of its terms all positive, while A remains unaltered. The body of the multiplication table is altered only in having all its signs made positive. If $a \neq 0$, the resulting matrix M defines a linear transformation on the six variables A_{ij}. It is known (author's *Linear Groups*, pp. 145-155) to leave invariant the quadratic function

$$F \equiv A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23},$$

whose square equals A (§3). Of several geometrical interpretations, the most elegant consists in regarding the A_{ij} as the six line-coödinates in ordinary space, so that $F \equiv 0$. The matrix M therefore defines a transformation of the straight lines in space. We may also interpret the equation $D=a$ (or $S=0$) as the totality of straight lines whose six line-coördinates are subject to a quadratic condition.